Available online at www.sciencedirect.com

SCIENCE DIRECT®

J. Differential Equations 219 (2005) 375–389

Journal of Differential Equations

www.elsevier.com/locate/jde

Homoclinic solutions for a class of the second order Hamiltonian systems

Marek Izydorek*,[1], Joanna Janczewska[1]

Department of Technical Physics and Applied Mathematics, Gdańsk University of Technology, Narutowicza 11/12, 80-952 Gdańsk, Poland

Received 19 August 2004

Available online on 19 September 2005

Abstract

We study the existence of homoclinic orbits for the second order Hamiltonian system $\ddot{q}+V_q(t,q)=f(t)$, where $q\in\mathbb{R}^n$ and $V\in C^1(\mathbb{R}\times\mathbb{R}^n,\mathbb{R})$, $V(t,q)=-K(t,q)+W(t,q)$ is T-periodic in t. A map K satisfies the "pinching" condition $b_1|q|^2\leqslant K(t,q)\leqslant b_2|q|^2$, W is superlinear at the infinity and f is sufficiently small in $L^2(\mathbb{R},\mathbb{R}^n)$. A homoclinic orbit is obtained as a limit of $2kT$-periodic solutions of a certain sequence of the second order differential equations.

MSC: 37J45; 58E05; 34C37; 70H05

Keywords: Homoclinic orbit; Hamiltonian system; Critical point

1. Introduction

In this paper, we shall be concerned with the existence of homoclinic orbits for the second order Hamiltonian system:

$$\ddot{q}+V_q(t,q)=f(t), \tag{HS}$$

* Corresponding author. Fax: +48 58 347 28 21.

E-mail addresses: izydorek@mifgate.pg.gda.pl (M. Izydorek), janczewska@mifgate.pg.gda.pl (J. Janczewska).

[1] Supported by Grant KBN no.1 P03A 042 29.

0022-0396/$ - see front matter
doi:10.1016/j.jde.2005.06.029

where $t \in \mathbb{R}$, $q \in \mathbb{R}^n$ and functions $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}^n$ satisfy:

(H_1) $V(t,q) = -K(t,q) + W(t,q)$, where $K, W : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ are C^1-maps, T-periodic with respect to t, $T > 0$,

(H_2) there are constants $b_1, b_2 > 0$ such that for all $(t,q) \in \mathbb{R} \times \mathbb{R}^n$

$$b_1|q|^2 \leqslant K(t,q) \leqslant b_2|q|^2,$$

(H_3) for all $(t,q) \in \mathbb{R} \times \mathbb{R}^n$, $K(t,q) \leqslant (q, K_q(t,q)) \leqslant 2K(t,q)$,

(H_4) $W_q(t,q) = o(|q|)$, as $|q| \to 0$ uniformly with respect to t,

(H_5) there is a constant $\mu > 2$ such that for every $t \in \mathbb{R}$ and $q \in \mathbb{R}^n \setminus \{0\}$

$$0 < \mu W(t,q) \leqslant (q, W_q(t,q)),$$

(H_6) $f : \mathbb{R} \to \mathbb{R}^n$ is a continuous and bounded function.

Here and subsequently, $(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the standard inner product in $\mathbb{R}^n$ and $|\cdot|$ is the induced norm.

We will say that a solution q of (HS) *is homoclinic* (to 0) if $q(t) \to 0$ as $t \to \pm\infty$. In addition, if $q \not\equiv 0$ then q is called *a nontrivial homoclinic solution*.

For each $k \in \mathbb{N}$, let $E_k := W^{1,2}_{2kT}(\mathbb{R}, \mathbb{R}^n)$, the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm

$$\|q\|_{E_k} := \left(\int_{-kT}^{kT} (|\dot{q}(t)|^2 + |q(t)|^2)\, dt \right)^{1/2}.$$

Furthermore, let $L^{\infty}_{2kT}(\mathbb{R}, \mathbb{R}^n)$ denote a space of $2kT$-periodic essentially bounded (measurable) functions from $\mathbb{R}$ into $\mathbb{R}^n$ equipped with the norm

$$\|q\|_{L^{\infty}_{2kT}} := \mathit{ess}\ \sup\{|q(t)| : t \in [-kT, kT]\}.$$

We begin with a result which is a direct consequence of estimations made by Rabinowitz in [12].

Proposition 1.1. *There is a positive constant C such that for each $k \in N$ and $q \in E_k$ the following inequality holds*:

$$\|q\|_{L^{\infty}_{2kT}} \leqslant C\|q\|_{E_k}. \tag{1}$$

One can easily show that the inequality (1) holds true with constant $C = \sqrt{2}$ if $T \geqslant \frac{1}{2}$ (see Fact 2.8).

Set $M := \sup\{W(t,q): t \in [0,T], |q| = 1\}$, $\bar{b}_1 := \min\{1, 2b_1\}$, $\bar{b}_2 := \max\{1, 2b_2\}$ and suppose that:

(H_7) $2M < \bar{b}_1$ and $(\int_{\mathbb{R}} |f(t)|^2 \, dt)^{1/2} \leqslant \frac{\beta}{2C}$, where $0 < \beta < \bar{b}_1 - 2M$ and C is a constant from Proposition 1.1.

We will prove the following theorem:

Theorem 1.2. *If the conditions* (H_1)–(H_7) *are satisfied then the system* (HS) *possesses a nontrivial homoclinic solution* $q \in W^{1,2}(\mathbb{R}, \mathbb{R}^n)$ *such that* $\dot{q}(t) \to 0$ *as* $t \to \pm\infty$.

In recent years several authors studied homoclinic orbits for Hamiltonian systems via critical point theory. In particular, the second order systems were considered in [1,3,5–7,11–13,16], and those of the first order in [4,8–10,14,15]. Our study is motivated by a paper of Rabinowitz [12] in which the existence of a nontrivial homoclinic solution for the second order Hamiltonian system

$$\ddot{q} + V_q(t,q) = 0$$

was proved. The function V considered by the author is of the form

$$V(t,q) = -\tfrac{1}{2}(L(t)q, q) + \overline{W}(t,q), \tag{2}$$

where L is a continuous T-periodic matrix valued function such that $L(t)$ is positive definite and symmetric for all $t \in [0,T]$, $\overline{W}$ satisfies (H_4) and (H_5). Let us note that conditions (H_2) and (H_3) are satisfied if $K(t,q) = \frac{1}{2}(L(t)q,q)$. On the other hand, one can easily check that if

$$K(t,x) = \begin{cases} \left(1 + \frac{1}{1+x^2}\right) x^2 & \text{for } x \geqslant 0, \\ \left(1 + \frac{2}{1+x^2}\right) x^2 & \text{for } x < 0 \end{cases}$$

and $W(t,x) = x^4$, where $t, x \in \mathbb{R}$, then $V(t,x) = -K(t,x) + W(t,x)$ cannot be represented in the form (2) with $\overline{W}$ satisfying (H_4), (H_5) while V satisfies conditions (H_1)–(H_5). Hence, our theorem extends the result from [12] even if $f(t) = 0$. It follows from our assumptions that $q(t) = 0$ is a solution of (HS) only if $f(t) = 0$. Therefore, if f is a nonzero function the existence of a homoclinic solution of (HS) implies its nontriviality.

Similarly to [12] a homoclinic solution of (HS) is obtained as a limit, as $k \to +\infty$, of a certain sequence of functions $q_k \in E_k$. However, in our approach, we consider a sequence of systems of differential equations:

$$\ddot{q} + V_q(t,q) = f_k(t), \tag{HS_k}$$

where for each $k \in \mathbb{N}$, $f_k: \mathbb{R} \to \mathbb{R}^n$ is a $2kT$-periodic extension of the restriction of f to the interval $[-kT, kT]$ and q_k is a $2kT$-periodic solution of (HS$_k$) obtained via the Mountain Pass Theorem.

Part of the difficulty in treating (HS) is caused by the fact that in order to get appropriate convergence of the sequence of approximative functions $\{q_k\}_{k\in\mathbb{N}}$ we need the constants ρ and α appearing in the condition (iii) of the Mountain Pass Theorem (see Theorem 2.5) to be independent of k.

2. Proof of Theorem 1.2

At first let us recall some properties of the function $W(t, q)$ from [12]. They all are necessary to the proof of Theorem 1.2.

Fact 2.1. *For every $t \in [0, T]$ the following inequalities hold*:

$$W(t, q) \leqslant W\left(t, \frac{q}{|q|}\right)|q|^{\mu} \quad \text{if } 0 < |q| \leqslant 1, \tag{3}$$

$$W(t, q) \geqslant W\left(t, \frac{q}{|q|}\right)|q|^{\mu} \quad \text{if } |q| \geqslant 1. \tag{4}$$

To prove this fact it suffices to show that for every $q \neq 0$ and $t \in [0, T]$ the function $(0, +\infty) \ni \zeta \to W(t, \zeta^{-1}q)\zeta^{\mu}$ is nonincreasing. It is an immediate consequence of (H$_5$).

Fact 2.2. *Set $m := \inf\{W(t, q): t \in [0, T], |q| = 1\}$. Then for every $\zeta \in \mathbb{R} \setminus \{0\}$ and $q \in E_k \setminus \{0\}$ we have*

$$\int_{-kT}^{kT} W(t, \zeta q(t))\, dt \geqslant m|\zeta|^{\mu} \int_{-kT}^{kT} |q(t)|^{\mu}\, dt - 2kTm. \tag{5}$$

Proof. Fix $\zeta \in \mathbb{R} \setminus \{0\}$ and $q \in E_k \setminus \{0\}$. Set $A_k = \{t \in [-kT, kT]: |\zeta q(t)| \leqslant 1\}$, and $B_k = \{t \in [-kT, kT]: |\zeta q(t)| \geqslant 1\}$. From (4) we obtain

$$\begin{aligned}
\int_{-kT}^{kT} W(t, \zeta q(t))\, dt &\geqslant \int_{B_k} W(t, \zeta q(t))\, dt \geqslant \int_{B_k} W\left(t, \frac{\zeta q(t)}{|\zeta q(t)|}\right)|\zeta q(t)|^{\mu}\, dt \\
&\geqslant m \int_{B_k} |\zeta q(t)|^{\mu}\, dt \\
&\geqslant m \int_{-kT}^{kT} |\zeta q(t)|^{\mu}\, dt - m \int_{A_k} |\zeta q(t)|^{\mu}\, dt \\
&\geqslant m|\zeta|^{\mu} \int_{-kT}^{kT} |q(t)|^{\mu}\, dt - 2kTm. \qquad \square
\end{aligned}$$

Fact 2.3. *Let* $Y\colon [0,+\infty) \to [0,+\infty)$ *be given as follows*: $Y(0) = 0$ *and*

$$Y(s) = \max_{\substack{t\in[0,T]\\ 0<|q|\leqslant s}} \frac{(q, W_q(t,q))}{|q|^2} \tag{6}$$

for $s > 0$. *Then* Y *is continuous*, *nondecreasing*, $Y(s) > 0$ *for* $s > 0$ *and* $Y(s) \to +\infty$ *as* $s \to +\infty$.

It is easy to verify this fact applying (H_4), (H_5) and (4).

Assumptions (H_4) and (H_5) imply that $W(t,q) = o(|q|^2)$ as $q \to 0$ uniformly for $t \in [0,T]$ and $W(t,0) = 0$, $W_q(t,0) = 0$. Moreover, from (H_2) we conclude that $K(t,0) = 0$, $K_q(t,0) = 0$.

Before we will prove Theorem 1.2, we have to introduce more notation and some necessary definitions. For each $k \in \mathbb{N}$, let $L^2_{2kT}(\mathbb{R},\mathbb{R}^n)$ denote the Hilbert space of $2kT$-periodic functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the norm $\|q\|_{L^2_{2kT}} = \left(\int_{-kT}^{kT} |q(t)|^2 dt\right)^{1/2}$. Let $f_k\colon \mathbb{R} \to \mathbb{R}^n$ be a $2kT$-periodic extension of $f_{|[-kT,kT]}$ onto $\mathbb{R}$. From (H_7) it follows that $\|f_k\|_{L^2_{2kT}} \leqslant \beta/2C$. Consider the second order Hamiltonian system:

$$\ddot{q} + V_q(t,q) = f_k(t). \tag{HS$_k$}$$

Let $\eta_k\colon E_k \to [0,+\infty)$ be given by

$$\eta_k(q) = \left(\int_{-kT}^{kT} \left[|\dot{q}(t)|^2 + 2K(t,q(t))\right] dt\right)^{1/2}. \tag{7}$$

By (H_2),

$$\bar{b}_1 \|q\|^2_{E_k} \leqslant \eta_k^2(q) \leqslant \bar{b}_2 \|q\|^2_{E_k}. \tag{8}$$

It is worth pointing out that if the function $K(t,q)$ is of the form $\frac{1}{2}(L(t)q,q)$ with a matrix valued function L satisfying the same conditions as in [12] then η_k determined by (7) is a norm in E_k equivalent to the norm $\|\cdot\|_{E_k}$. Let $I_k\colon E_k \to \mathbb{R}$ be defined by

$$\begin{aligned} I_k(q) &= \int_{-kT}^{kT} \left[\frac{1}{2}|\dot{q}(t)|^2 - V(t,q(t))\right] dt + \int_{-kT}^{kT} (f_k(t), q(t))\, dt \\ &= \frac{1}{2}\eta_k^2(q) - \int_{-kT}^{kT} W(t,q(t))\, dt + \int_{-kT}^{kT} (f_k(t), q(t))\, dt. \end{aligned} \tag{9}$$

Then $I_k \in C^1(E_k,\mathbb{R})$ and it is easy to check that

$$I_k'(q)v = \int_{-kT}^{kT} \left[(\dot{q}(t), \dot{v}(t)) - (V_q(t,q(t)), v(t))\right] dt + \int_{-kT}^{kT} (f_k(t), v(t))\, dt, \tag{10}$$

and

$$I'_k(q)q \leqslant \eta_k^2(q) - \int_{-kT}^{kT} (W_q(t, q(t)), q(t))\, dt + \int_{-kT}^{kT} (f_k(t), q(t))\, dt. \tag{11}$$

Moreover, it is clear that critical points of I_k are classical $2kT$-periodic solutions of (HS_k).

We have divided the proof of Theorem 1.2 into a sequence of lemmas.

Lemma 2.4. *If V and f satisfy* (H_1)–(H_7) *then for every $k \in \mathbb{N}$ the system* (HS_k) *possesses a $2kT$-periodic solution.*

We will obtain a critical point of I_k by the use of a standard version of the Mountain Pass Theorem (see [2]). It provides the minimax characterisation for the critical value which is important for what follows. Therefore, we state this theorem precisely.

Theorem 2.5 (*see Ambrosetti and Rabinowitz [2]*). *Let E be a real Banach space and $I: E \to \mathbb{R}$ be a C^1-smooth functional. If I satisfies the following conditions*:

(i) $I(0) = 0$,
(ii) *every sequence $\{u_j\}_{j\in\mathbb{N}}$ in E such that $\{I(u_j)\}_{j\in\mathbb{N}}$ is bounded in $\mathbb{R}$ and $I'(u_j) \to 0$ in E^*, as $j \to +\infty$, contains a convergent subsequence* (*the Palais-Smale condition*),
(iii) *there exist constants ϱ, $\alpha > 0$ such that $I_{|\partial B_\varrho(0)} \geqslant \alpha$*,
(iv) *there exists $e \in E \setminus \overline{B}_\varrho(0)$ such that $I(e) \leqslant 0$*,

where $B_\varrho(0)$ is an open ball in E of radius ϱ centred at 0, *then I possesses a critical value $c \geqslant \alpha$ given by*

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0, 1], E): g(0) = 0, \;\; g(1) = e\}.$$

Proof of Lemma 2.4. In our case it is clear that $I_k(0) = 0$. We show that I_k satisfies the Palais-Smale condition. Assume that $\{u_j\}_{j\in\mathbb{N}}$ in E_k is a sequence such that $\{I_k(u_j)\}_{j\in\mathbb{N}}$ is bounded and $I'_k(u_j) \to 0$ as $j \to +\infty$. Then there exists a constant $C_k > 0$ such that

$$|I_k(u_j)| \leqslant C_k, \qquad \|I'_k(u_j)\|_{E_k^*} \leqslant C_k \tag{12}$$

for every $j \in \mathbb{N}$. We first prove that $\{u_j\}_{j\in\mathbb{N}}$ is bounded. By (9) and (H_5),

$$\eta_k^2(u_j) \leqslant 2I_k(u_j) + \frac{2}{\mu}\int_{-kT}^{kT} (W_q(t,u_j(t)), u_j(t))\,dt - 2\int_{-kT}^{kT} (f_k(t), u_j(t))\,dt. \tag{13}$$

From (13) and (11) we obtain

$$\left(1-\frac{2}{\mu}\right)\eta_k^2(u_j) \leqslant 2I_k(u_j) - \frac{2}{\mu}I_k'(u_j)u_j - \left(2-\frac{2}{\mu}\right)\int_{-kT}^{kT} (f_k(t), u_j(t))\,dt. \tag{14}$$

From (14) and (8) it follows that

$$\left(1-\frac{2}{\mu}\right)\bar{b}_1\|u_j\|_{E_k}^2 \leqslant 2I_k(u_j) + \left(\frac{2}{\mu}\|I_k'(u_j)\|_{E_k^*} + \left(2-\frac{2}{\mu}\right)\|f_k\|_{L^2_{2kT}}\right)\|u_j\|_{E_k}. \tag{15}$$

Combining (15) with (H_7) and (12) we get

$$\left(1-\frac{2}{\mu}\right)\bar{b}_1\|u_j\|_{E_k}^2 - \left(\frac{2C_k}{\mu} + \left(2-\frac{2}{\mu}\right)\frac{\beta}{2C}\right)\|u_j\|_{E_k} - 2C_k \leqslant 0. \tag{16}$$

Since $\mu > 2$, (16) shows that $\{u_j\}_{j\in\mathbb{N}}$ is bounded in E_k. Going if necessary to a subsequence, we can assume that there exists $u \in E_k$ such that $u_j \rightharpoonup u$, as $j \to +\infty$, in E_k, which implies $u_j \to u$ uniformly on $[-kT, kT]$. Hence $\left(I_k'(u_j) - I_k'(u)\right)(u_j - u) \to 0$, $\|u_j - u\|_{L^2_{2kT}} \to 0$ and

$$\int_{-kT}^{kT} (V_q(t,u_j(t)) - V_q(t,u(t)), u_j(t) - u(t))\,dt \to 0,$$

as $j \to +\infty$. Moreover, an easy computation shows that

$$\left(I_k'(u_j) - I_k'(u)\right)(u_j - u) = \|\dot{u}_j - \dot{u}\|_{L^2_{2kT}}^2 - \int_{-kT}^{kT} (V_q(t,u_j(t)) - V_q(t,u(t)), u_j(t) - u(t))\,dt,$$

and so $\|\dot{u}_j - \dot{u}\|_{L^2_{2kT}}^2 \to 0$. Consequently, $\|u_j - u\|_{E_k} \to 0$.

We now show that there exist constants ϱ, $\alpha > 0$ independent of k such that every I_k satisfies the assumption (iii) of Theorem 2.5 with these constants. Assume that $0 < \|q\|_{L^\infty_{2kT}} \leqslant 1$. By (3) we have

$$\begin{aligned}\int_{-kT}^{kT} W(t, q(t))\, dt &\leqslant \int_{-kT}^{kT} W\left(t, \frac{q(t)}{|q(t)|}\right) |q(t)|^\mu\, dt \\ &\leqslant M \int_{-kT}^{kT} |q(t)|^2\, dt \leqslant M\|q\|^2_{E_k},\end{aligned}$$

and, in consequence, combining this with (8) and (H_7) we obtain

$$\begin{aligned} I_k(q) &\geqslant \frac{1}{2}\bar{b}_1\|q\|^2_{E_k} - M\|q\|^2_{E_k} - \|f_k\|_{L^2_{2kT}}\|q\|_{L^2_{2kT}} \\ &\geqslant \frac{1}{2}\bar{b}_1\|q\|^2_{E_k} - M\|q\|^2_{E_k} - \frac{\beta}{2C}\|q\|_{E_k} \\ &= \frac{1}{2}(\bar{b}_1 - \beta - 2M)\|q\|^2_{E_k} + \frac{\beta}{2}\|q\|^2_{E_k} - \frac{\beta}{2C}\|q\|_{E_k}. \end{aligned} \tag{17}$$

Note that (H_7) implies $\bar{b}_1 - \beta - 2M > 0$. Set

$$\varrho = \frac{1}{C}, \qquad \alpha = \frac{\bar{b}_1 - \beta - 2M}{2C^2}.$$

By (1), if $\|q\|_{E_k} = \varrho$ then $0 < \|q\|_{L^\infty_{2kT}} \leqslant 1$ and (17) gives $I_k(q) \geqslant \alpha$.

It remains to prove that for every $k \in \mathbb{N}$ there exists $e_k \in E_k$ such that $\|e_k\|_{E_k} > \varrho$ and $I_k(e_k) \leqslant 0$. By the use of (5), (9) and (8) we have that for every $\zeta \in \mathbb{R} \setminus \{0\}$ and $q \in E_k \setminus \{0\}$ the following inequality holds:

$$\begin{aligned} I_k(\zeta q) &\leqslant \frac{\bar{b}_2\zeta^2}{2}\|q\|^2_{E_k} - m|\zeta|^\mu \int_{-kT}^{kT} |q(t)|^\mu\, dt \\ &\quad + |\zeta| \cdot \|f_k\|_{L^2_{2kT}}\|q\|_{L^2_{2kT}} + 2kTm. \end{aligned} \tag{18}$$

Take $Q \in E_1$ such that $Q(\pm T) = 0$. Since $\mu > 2$ and $m > 0$, (18) implies that there exists $\xi \in \mathbb{R} \setminus \{0\}$ such that $\|\xi Q\|_{E_1} > \varrho$ and $I_1(\xi Q) < 0$. Set $e_1(t) = \xi Q(t)$ and

$$e_k(t) = \begin{cases} e_1(t) & \text{for } |t| \leqslant T, \\ 0 & \text{for } T < |t| \leqslant kT \end{cases} \tag{19}$$

for $k > 0$. Then $e_k \in E_k$, $\|e_k\|_{E_k} = \|e_1\|_{E_1} > \varrho$ and $I_k(e_k) = I_1(e_1) < 0$ for every $k \in \mathbb{N}$. By Theorem 2.5, I_k possesses a critical value $c_k \geqslant \alpha$ given by

$$c_k = \inf_{g \in \Gamma_k} \max_{s \in [0,1]} I_k(g(s)), \tag{20}$$

where

$$\Gamma_k = \{g \in C([0,1], E_k) : g(0) = 0,\ \ g(1) = e_k\}.$$

Hence, for every $k \in \mathbb{N}$, there is $q_k \in E_k$ such that

$$I_k(q_k) = c_k, \qquad I_k'(q_k) = 0. \tag{21}$$

The function q_k is a desired classical $2kT$-periodic solution of (HS$_k$). Since $c_k > 0$, q_k is a nontrivial solution even if $f_k(t) = 0$. $\square$

Let $C^p_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$, where $p \in \mathbb{N} \cup \{0\}$, denote the space of C^p functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ under the topology of almost uniformly convergence of functions and all derivatives up to the order p. Using the Arzelà-Ascoli theorem we prove what follows.

Lemma 2.6. *Let $\{q_k\}_{k\in\mathbb{N}}$ be the sequence given by* (21). *There exist an increasing function $\varphi: \mathbb{N} \to \mathbb{N}$ and a C^1 function $q_0: \mathbb{R} \to \mathbb{R}^n$ such that $q_{\varphi(k)} \to q_0$, as $k \to +\infty$, in $C^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$.*

Proof. The first step in the proof is to show that the sequences $\{c_k\}_{k\in\mathbb{N}}$ and $\{\|q_k\|_{E_k}\}_{k\in\mathbb{N}}$ are bounded. For every $k \in \mathbb{N}$, let $g_k: [0,1] \to E_k$ be a curve given by $g_k(s) = se_k$, where e_k is determined by (19). Then $g_k \in \Gamma_k$ and $I_k(g_k(s)) = I_1(g_1(s))$ for all $k \in \mathbb{N}$ and $s \in [0,1]$. Therefore, by (20),

$$c_k \leqslant \max_{s\in[0,1]} I_1(g_1(s)) \equiv M_0 \tag{22}$$

independently of $k \in \mathbb{N}$. As $I_k'(q_k) = 0$, we receive from (9), (11) and (H$_5$) that

$$\begin{aligned} c_k &= I_k(q_k) - \frac{1}{2} I_k'(q_k) q_k \\ &\geqslant \left(\frac{\mu}{2} - 1\right) \int_{-kT}^{kT} W(t, q_k(t))\, dt + \frac{1}{2} \int_{-kT}^{kT} (f_k(t), q_k(t))\, dt, \end{aligned}$$

and hence

$$\int_{-kT}^{kT} W(t, q_k(t))\, dt \leqslant \frac{1}{\mu - 2} \left(2c_k - \int_{-kT}^{kT} (f_k(t), q_k(t))\, dt \right).$$

Combining the above with (8), (9) and (22) we have

$$\bar{b}_1 \|q_k\|^2_{E_k} \leqslant \frac{2\mu M_0}{\mu - 2} + \frac{2\mu - 2}{\mu - 2} \|f_k\|_{L^2_{2kT}} \|q_k\|_{L^2_{2kT}},$$

and, in consequence, by (H_7)

$$\bar{b}_1 \|q_k\|_{E_k}^2 - \frac{\beta(\mu-1)}{C(\mu-2)} \|q_k\|_{E_k} - \frac{2\mu M_0}{\mu-2} \leqslant 0. \tag{23}$$

Since $\bar{b}_1 > 0$ and all coefficients of (23) are independent of k, we see that there is $M_1 > 0$ independent of k such that

$$\|q_k\|_{E_k} \leqslant M_1. \tag{24}$$

We now observe that the sequences $\{q_k\}_{k\in\mathbb{N}}$, $\{\dot{q}_k\}_{k\in\mathbb{N}}$ and $\{\ddot{q}_k\}_{k\in\mathbb{N}}$ are uniformly bounded. By (1),

$$\|q_k\|_{L^\infty_{2kT}} \leqslant C M_1 \equiv M_2 \tag{25}$$

for every $k \in \mathbb{N}$. Since q_k satisfies (HS_k), if $t \in [-kT, kT]$ we have

$$|\ddot{q}_k(t)| \leqslant |f_k(t)| + |V_q(t, q_k(t))| = |f(t)| + |V_q(t, q_k(t))|.$$

Therefore (25), (H_1) and (H_6) imply that there is $M_3 > 0$ independent of k such that

$$\|\ddot{q}_k\|_{L^\infty_{2kT}} \leqslant M_3. \tag{26}$$

From the Mean Value Theorem it follows that for every $k \in \mathbb{N}$ and $t \in \mathbb{R}$ there exists $\tau_k \in [t-1, t]$ such that

$$\dot{q}_k(\tau_k) = \int_{t-1}^{t} \dot{q}_k(s)\, ds = q_k(t) - q_k(t-1).$$

In consequence, combining the above with (25) and (26)

$$\begin{aligned} |\dot{q}_k(t)| &= \left| \int_{\tau_k}^{t} \ddot{q}_k(s)\, ds + \dot{q}_k(\tau_k) \right| \\ &\leqslant \int_{t-1}^{t} |\ddot{q}_k(s)|\, ds + |q_k(t) - q_k(t-1)| \leqslant M_3 + 2M_2 \equiv M_4, \end{aligned}$$

and hence for every $k \in \mathbb{N}$

$$\|\dot{q}_k\|_{L^\infty_{2kT}} \leqslant M_4. \tag{27}$$

The task is now to show that $\{q_k\}_{k\in\mathbb{N}}$ and $\{\dot{q}_k\}_{k\in\mathbb{N}}$ are equicontinuous. Of course, it suffices to prove that both sequences satisfy the Lipschitz condition with some constants independent of k. Let $k \in \mathbb{N}$ and $t,\ t_0 \in \mathbb{R}$. Then

$$|q_k(t) - q_k(t_0)| = \left|\int_{t_0}^{t} \dot{q}_k(s)\,ds\right| \leqslant \left|\int_{t_0}^{t} |\dot{q}_k(s)|\,ds\right| \leqslant M_4|t-t_0|,$$

by (27), and analogously,

$$|\dot{q}_k(t) - \dot{q}_k(t_0)| \leqslant M_3|t-t_0|,$$

by (26). Since $\{q_k\}_{k\in\mathbb{N}}$ and $\{\dot{q}_k\}_{k\in\mathbb{N}}$ are bounded in $L^{\infty}_{2kT}(\mathbb{R}, \mathbb{R}^n)$ and equicontinuous, we obtain the existence of a subsequence $\{q_{\varphi(k)}\}_{k\in\mathbb{N}}$ convergent to a certain q_0 in $C^1_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ by using the Arzelà-Ascoli theorem. $\square$

Our next goal is to show that q_0 is the desired homoclinic solution of (HS). For this purpose, we need the following observations.

Fact 2.7. *Let $q\colon \mathbb{R} \to \mathbb{R}^n$ be a continuous mapping. If a weak derivative $\dot{q}\colon \mathbb{R} \to \mathbb{R}^n$ is continuous at t_0, then q is differentiable at t_0 and*

$$\lim_{t\to t_0} \frac{q(t) - q(t_0)}{t - t_0} = \dot{q}(t_0).$$

Proof. Fix $\varepsilon > 0$. By the assumption, there exists $\delta > 0$ such that for every $t \in \mathbb{R}$, if $|t - t_0| < \delta$ then $|\dot{q}(t) - \dot{q}(t_0)| < \varepsilon$. Hence

$$\left|\frac{q(t) - q(t_0)}{t - t_0} - \dot{q}(t_0)\right| = \left|\frac{\int_{t_0}^{t} (\dot{q}(s) - \dot{q}(t_0))\,ds}{t - t_0}\right| \leqslant \frac{\int_{t_0}^{t} |\dot{q}(s) - \dot{q}(t_0)|\,ds}{|t - t_0|} \leqslant \varepsilon$$

provided that $0 < |t - t_0| < \delta$. $\square$

Let $L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$ denote the space of functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ locally square integrable.

Fact 2.8. *Let $q\colon \mathbb{R} \to \mathbb{R}^n$ be a continuous mapping such that $\dot{q} \in L^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$. For every $t \in \mathbb{R}$ the following inequality holds*:

$$|q(t)| \leqslant \sqrt{2}\left(\int_{t-1/2}^{t+1/2} \left(|q(s)|^2 + |\dot{q}(s)|^2\right) ds\right)^{1/2}. \tag{28}$$

Proof. Fix $t \in \mathbb{R}$. For every $\tau \in \mathbb{R}$,

$$|q(t)| \leqslant |q(\tau)| + \left| \int_{\tau}^{t} \dot{q}(s)\, ds \right|. \tag{29}$$

Integrating (29) over $[t - \frac{1}{2}, t + \frac{1}{2}]$ and using the Hölder inequality we obtain

$$\begin{aligned} |q(t)| &\leqslant \int_{t-1/2}^{t+1/2} \left(|q(\tau)| + \left| \int_{\tau}^{t} \dot{q}(s)\, ds \right| \right) d\tau \\ &\leqslant \left(\int_{t-1/2}^{t+1/2} \left(|q(\tau)| + \left| \int_{\tau}^{t} \dot{q}(s)\, ds \right| \right)^2 d\tau \right)^{1/2} \\ &\leqslant \left(2 \int_{t-1/2}^{t+1/2} \left(|q(\tau)|^2 + \left| \int_{\tau}^{t} \dot{q}(s)\, ds \right|^2 \right) d\tau \right)^{1/2} \\ &\leqslant \sqrt{2} \left(\int_{t-1/2}^{t+1/2} |q(\tau)|^2\, d\tau + \int_{t-1/2}^{t+1/2} |\dot{q}(s)|^2\, ds \right)^{1/2}. \qquad \square \end{aligned}$$

Lemma 2.9. *The function* q_0 *determined by Lemma* 2.6 *is the desired homoclinic solution of* (HS).

Proof. The proof will be divided into four steps.

Step 1: We show that q_0 is a solution of (HS). For every $k \in \mathbb{N}$ and $t \in \mathbb{R}$ we have

$$\ddot{q}_{\varphi(k)}(t) = f_{\varphi(k)}(t) - V_q(t, q_{\varphi(k)}(t)). \tag{30}$$

Since $q_{\varphi(k)} \to q_0$ and $f_{\varphi(k)} \to f$ almost uniformly on $\mathbb{R}$, we obtain that $\ddot{q}_{\varphi(k)} \to w$ almost uniformly on $\mathbb{R}$, where $w(t) = f(t) - V_q(t, q_0(t))$. Fix $a,\ b \in \mathbb{R}$ such that $a < b$. There is $k_0 \in \mathbb{N}$ such that for every $k \geqslant k_0$ and $t \in [a, b]$, (30) becomes

$$\ddot{q}_{\varphi(k)}(t) = f(t) - V_q(t, q_{\varphi(k)}(t)).$$

Hence, if $k \geqslant k_0$ then the restriction of $\ddot{q}_{\varphi(k)}$ onto $[a, b]$ is continuous. From Fact 2.7 it follows that $\ddot{q}_{\varphi(k)}$ is a derivative of $\dot{q}_{\varphi(k)}$ in (a, b) for every $k \geqslant k_0$. Since $\ddot{q}_{\varphi(k)} \to w$ and $\dot{q}_{\varphi(k)} \to \dot{q}_0$ almost uniformly on $\mathbb{R}$, we have $w = \ddot{q}_0$ in (a, b). By the above, we conclude that $w = \ddot{q}_0$ in $\mathbb{R}$ and q_0 satisfies (HS). Moreover, note that we have actually proved that $\{q_{\varphi(k)}\}_{k \in \mathbb{N}}$ converges to q_0 in the topology of $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^n)$.

Step 2: We prove that $q_0(t) \to 0$, as $t \to \pm\infty$. We have

$$\int_{-\infty}^{+\infty} \left(|q_0(t)|^2 + |\dot{q}_0(t)|^2\right) dt = \lim_{i\to+\infty} \int_{-iT}^{iT} \left(|q_0(t)|^2 + |\dot{q}_0(t)|^2\right) dt$$
$$= \lim_{i\to+\infty} \lim_{k\to+\infty} \int_{-iT}^{iT} \left(|q_{\varphi(k)}(t)|^2 + |\dot{q}_{\varphi(k)}(t)|^2\right) dt.$$

Clearly, for every $i \in \mathbb{N}$ there exists $k_i \in \mathbb{N}$ such that for all $k \geqslant k_i$ we have

$$\int_{-iT}^{iT} \left(|q_{\varphi(k)}(t)|^2 + |\dot{q}_{\varphi(k)}(t)|^2\right) dt \leqslant \|q_{\varphi(k)}\|^2_{E_{\varphi(k)}} \leqslant M_1^2,$$

by (24). Letting $k \to +\infty$, we get

$$\int_{-iT}^{iT} \left(|q_0(t)|^2 + |\dot{q}_0(t)|^2\right) dt \leqslant M_1^2,$$

and now, letting $i \to +\infty$, we have

$$\int_{-\infty}^{+\infty} \left(|q_0(t)|^2 + |\dot{q}_0(t)|^2\right) dt \leqslant M_1^2,$$

and so

$$\int_{|t|\geqslant r} \left(|q_0(t)|^2 + |\dot{q}_0(t)|^2\right) dt \to 0, \tag{31}$$

as $r \to +\infty$. Combining (31) with (28) we receive our claim.

Step 3: We now show that $\dot{q}_0(t) \to 0$, as $t \to \pm\infty$. To do this, observe that

$$|\dot{q}_0(t)|^2 \leqslant 2 \int_{t-1/2}^{t+1/2} \left(|q_0(s)|^2 + |\dot{q}_0(s)|^2\right) ds + 2 \int_{t-1/2}^{t+1/2} |\ddot{q}_0(s)|^2 \, ds, \tag{32}$$

by (28). Since we have (31) and (32) it suffices to prove that

$$\int_r^{r+1} |\ddot{q}_0(s)|^2 \, ds \to 0, \tag{33}$$

as $r \to \pm\infty$. By (HS) we obtain

$$\int_r^{r+1} |\ddot{q}_0(s)|^2 \, ds = \int_r^{r+1} \left(|V_q(s, q_0(s))|^2 + |f(s)|^2\right) ds$$
$$-2 \int_r^{r+1} (V_q(s, q_0(s)), f(s)) \, ds.$$

Since $V_q(t,0)=0$ for all $t\in\mathbb{R}$, $q_0(t)\to 0$, as $t\to\pm\infty$ and $\int_r^{r+1}|f(s)|^2\,ds\to 0$, as $r\to\pm\infty$, (33) follows.

Step 4: In the end, we have to show that if $f\equiv 0$ then $q_0\not\equiv 0$. For this purpose, as Rabinowitz we use the properties of Y given by (6). The definition of Y implies

$$\int_{-kT}^{kT}(W_q(t,q_k(t)),q_k(t))\,dt\leqslant Y(\|q_k\|_{L^\infty_{2kT}})\|q_k\|^2_{E_k} \tag{34}$$

for every $k\in\mathbb{N}$. Since $I_k'(q_k)q_k=0$, (10) gives

$$\int_{-kT}^{kT}(W_q(t,q_k(t)),q_k(t))\,dt=\int_{-kT}^{kT}|\dot q_k(t)|^2\,dt+\int_{-kT}^{kT}(K_q(t,q_k(t)),q_k(t))\,dt. \tag{35}$$

Substituting (35) into (34), and next applying (H_3) and (H_2) we obtain

$$Y(\|q_k\|_{L^\infty_{2kT}})\|q_k\|^2_{E_k}\geqslant \min\{1,b_1\}\|q_k\|^2_{E_k},$$

and hence

$$Y(\|q_k\|_{L^\infty_{2kT}})\geqslant \min\{1,b_1\}>0. \tag{36}$$

The remainder of the proof is the same as in [12]. If $\|q_k\|_{L^\infty_{2kT}}\to 0$, as $k\to+\infty$, we would have $Y(0)\geqslant\min\{1,b_1\}>0$, a contradiction. Thus there is $\gamma>0$ such that

$$\|q_k\|_{L^\infty_{2kT}}\geqslant\gamma \tag{37}$$

for every $k\in\mathbb{N}$. Clearly, $q_k(t+jT)$ is a $2kT$-periodic solution of (HS_k) for every $j\in\mathbb{Z}$. By replacing earlier, if necessary, q_k by $q_k(t+jT)$ for some $j\in[-k,k]\cap\mathbb{Z}$, one can assume that the maximum of q_k occurs in $[-T,T]$. Suppose, contrary to our claim, that $q_0\equiv 0$. Then, by Lemma 2.6,

$$\|q_{\varphi(k)}\|_{L^\infty_{2\varphi(k)T}}=\max_{t\in[-T,T]}|q_{\varphi(k)}(t)|\to 0,$$

which contradicts (37). $\square$

References

[1] A. Ambrosetti, V. Coti Zelati, Multiple homoclinic orbits for a class of conservative systems, Rend. Sem. Mat. Univ. Padova 89 (1993) 177–194.

[2] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.

[3] P.C. Carrião, O.H. Miyagaki, Existence of homoclinic solutions for a class of time-dependent Hamiltonian systems, J. Math. Anal. Appl. 230 (1999) 157–172.
[4] V. Coti Zelati, I. Ekeland, E. Séré, A variational approach to homoclinic orbits in Hamiltonian systems, Math. Ann. 228 (1990) 133–160.
[5] V. Coti Zelati, P.H. Rabinowitz, Homoclinic orbits for second order Hamiltonian systems possessing superquadratic potentials, J. Amer. Math. Soc. 4 (1991) 693–727.
[6] Y. Ding, Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems, Nonlinear Anal. 25 (1995) 1095–1113.
[7] Y. Ding, M. Girardi, Periodic and homoclinic solutions to a class of Hamiltonian systems with the potentials changing sign, J. Math. Anal. Appl. 189 (1995) 585–601.
[8] Y. Ding, S. Li, Homoclinic orbits for first order Hamiltonian systems, J. Math. Anal. Appl. 189 (1995) 585–601.
[9] Y. Ding, M. Willem, Homoclinic orbits of a Hamiltonian system, Z. Angew. Math. Phys. 50 (1999) 759–778.
[10] H. Hofer, K. Wysocki, First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems, Math. Ann. 228 (1990) 483–503.
[11] W. Omana, M. Willem, Homoclinic orbits for a class of Hamiltonian systems, Differential Integral Equations 5 (1992) 1115–1120.
[12] P.H. Rabinowitz, Homoclinic orbits for a class of Hamiltonian systems, Proc. Roy. Soc. Edinburgh 114A (1990) 33–38.
[13] P.H. Rabinowitz, K. Tanaka, Some results on connecting orbits for a class of Hamiltonian systems, Math. Z. 206 (1991) 472–499.
[14] E. Séré, Existence of infinitely many homoclinic orbits in Hamiltonian systems, Math. Z. 209 (1993) 561–590.
[15] A. Szulkin, W. Zou, Homoclinic orbits for asymptotically linear Hamiltonian systems, J. Funct. Anal. 187 (2001) 25–41.
[16] K. Tanaka, Homoclinic orbits for a singular second order Hamiltonian system, Ann. Inst. H. Poincaré 7 (5) (1990) 427–438.